Digital Logic

CMSC 313

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Binary Logic definitions

- Binary variables take on one of two values.
- Logical operators operate on binary values and binary variables.
- Basic logical operators are the logic functions AND, OR and NOT.
- Logic gates implement logic functions.
- Boolean Algebra: a useful mathematical system for specifying and transforming logic functions.
- We study Boolean algebra as foundation for designing and analyzing digital systems!

Binary Variables

- **Just like a regular variable, except takes only two values**
- **Recall that the two binary values have different names:**
  - **True/False**
  - **On/Off**
  - **Yes/No**
  - **1/0**
- **We use 1 and 0 to denote the two values.**

Boolean Operators

- **AND**
  - **denoted by a dot (·) or two variables immediately next to each other.**
  - **Y=A·B is read "Y is equal to A AND B."**
- **OR**
  - **denoted by a plus (+).**
  - **z = x + y is read "z is equal to x OR y."**
- **NOT**
  - **denoted by an overbar (¯), a single quote mark (') after, or (~) before the variable.**
  - $X = \bar{A}$ **is read "X is equal to NOT A."**
- **XOR**
  - **Denoted by a plus with a circle (⊕)**

Truth Tables

AND

X	Y	$X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

OR

X	Y	$X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT

X	$\overline{X}$
0	0
0	1

NAND

X	Y	$\overline{X \cdot Y}$
0	0	1
0	1	1
1	0	1
1	1	0

NOR

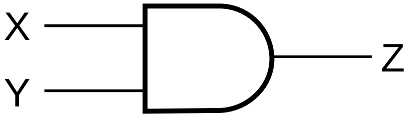
X	Y	$\overline{X+Y}$
0	0	1
0	1	1
1	0	1
1	1	0

XOR

X	Y	$X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

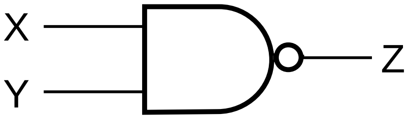
Gate Symbols

- Logic gates have special symbols:



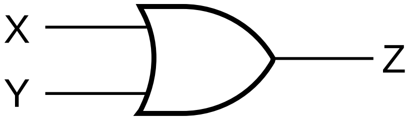
$XY = Z$

AND gate



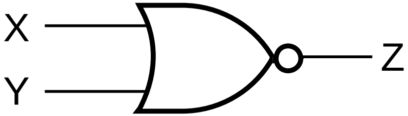
$\overline{XY} = Z$

NAND gate



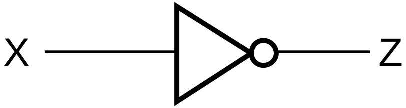
$X + Y = Z$

OR gate



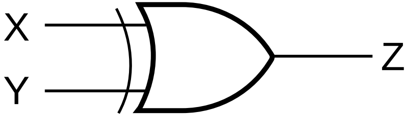
$\overline{X + Y} = Z$

NOR gate



$\bar{X} = Z$

NOT gate or inverter



$X \oplus Y = Z$

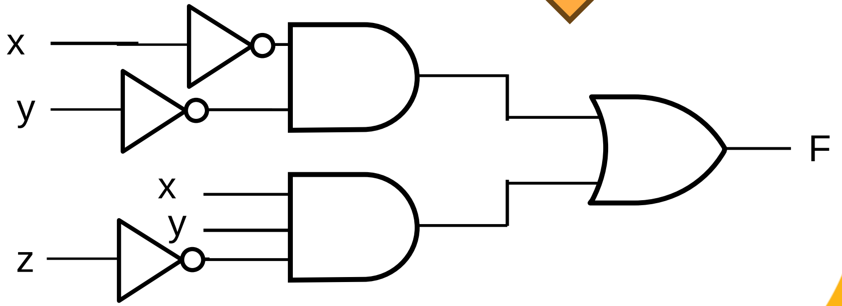
XOR gate

Representation

- Boolean equations, truth tables and logic diagrams describe the same function!
- Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

$$F = \bar{x}\bar{y} + xy\bar{z}$$



Order of Operations

- Order:
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- Consequence: Parentheses appear around OR expressions
- Example: $F = A(B + C)(C + \overline{D})$
  1. $\overline{D}$: not
  2. $B + C$, $C + \overline{D}$: parenthesis
  3. $A(B + C)(C + \overline{D})$: and

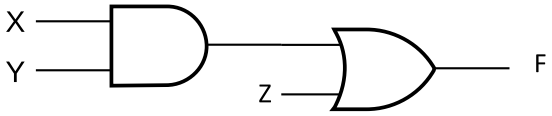
Boolean Function Evaluation

$$F1 = xy\bar{z}$$
$$F2 = x + \bar{y}z$$
$$F3 = \bar{x}\,\bar{y}\,\bar{z} + \bar{x}yz + x\bar{y}$$
$$F4 = x\bar{y} + \bar{x}z$$

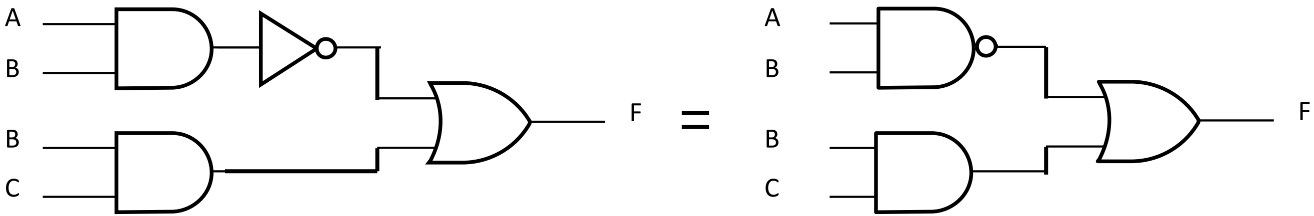
x	y	z	F1	F2	F3	F4
0	0	0	0	0	1	0
0	0	1	0	1	0	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

Logic Circuit Examples

- Example 1: Draw the circuit XY+Z = F

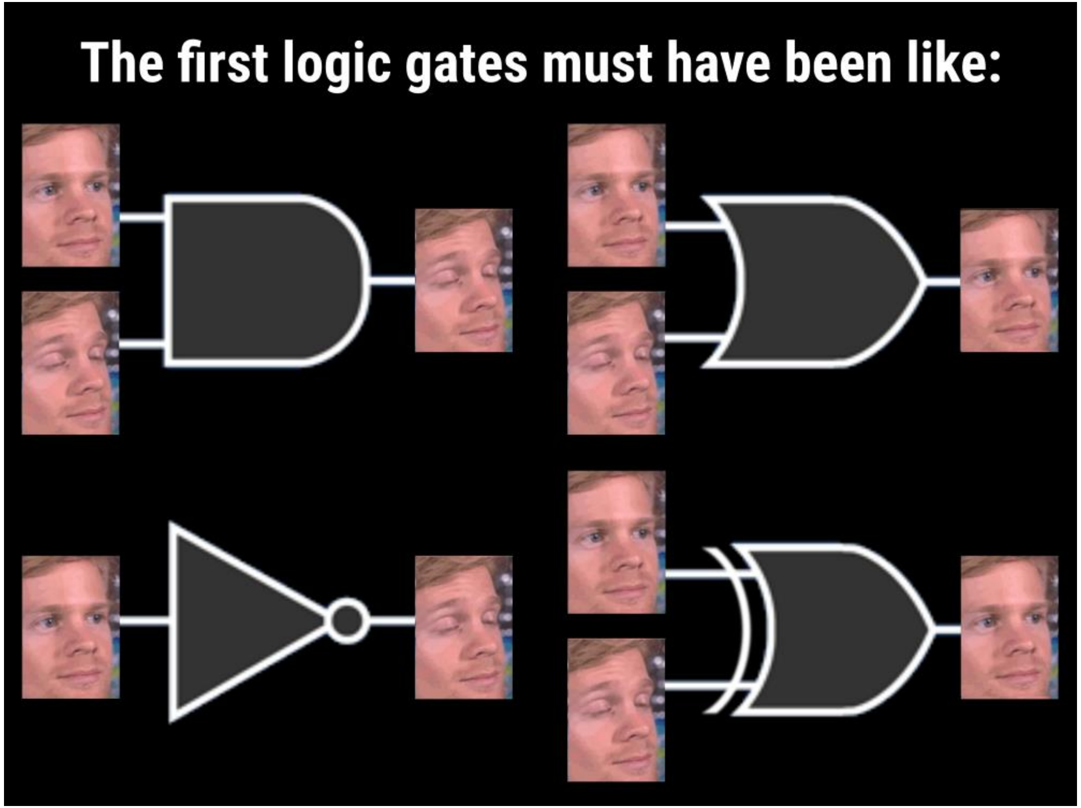


- Example 2: Find the Boolean expression



$\overline{AB} + BC = F$

Meme

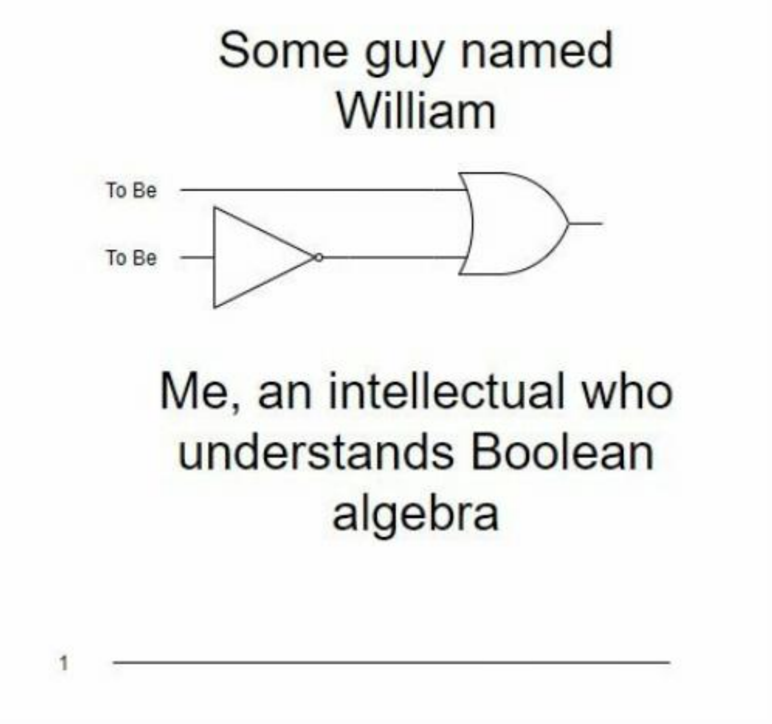


Boolean Algebra Identities

1.	$X + 0 = X$	2.	$X \cdot 1 = X$	Identity
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	Null Element
5.	$X + X = X$	6.	$X \cdot X = X$	Indempotence
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	Complement
9.	$\overline{\overline{X}} = X$			Involution
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Note: These are grouped as duals. The **dual** of an algebraic expression is obtained by interchanging + and · and interchanging 0's and 1's.

Moar memes



How inefficient of him

X + ~X = 1
Complement

Example 1: Boolean Algebraic Proof

- $A + A \cdot B = A$ (Absorption Theorem)

Proof Steps	Justification (identity or theorem)
$A + A \cdot B$	
$= A \cdot 1 + A \cdot B$	$X = X \cdot 1$ (Identity)
$= A \cdot (1 + B)$	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$ (Distributive Law)
$= A \cdot 1$	$1 + X = 1$ (Null element)
$= A$	$X \cdot 1 = X$ (Identity)

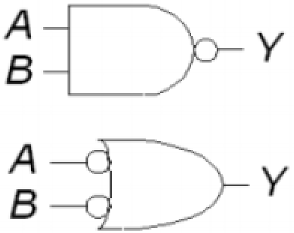
Example 2: Boolean Algebraic Proofs

- $AB + \bar{A}C + BC = AB + \bar{A}C$ (Consensus Theorem)

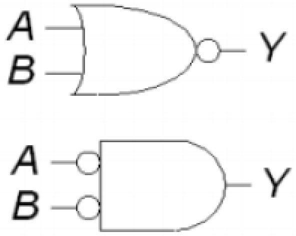
Proof Steps	Justification (identity or theorem)
$AB + \bar{A}C + BC$	
$= AB + \bar{A}C + 1 \cdot BC$	(identity)
$= AB + \bar{A}C + (A + \bar{A}) \cdot BC$	(complement)
$= AB + \bar{A}C + ABC + \bar{A}BC$	(Distributive Law)
$= AB + ABC + \bar{A}C + \bar{A}BC$	(Commutative Law)
$= AB \cdot 1 + ABC + \bar{A}C \cdot 1 + \bar{A}C \cdot B$	(identity, Commutative Law)
$= AB(1 + C) + \bar{A}C(1 + B)$	(Distributive Law)
$= AB \cdot 1 + \bar{A}C \cdot 1 = AB + \bar{A}C$	(identity)

DeMorgan's Law

$$\overline{A \bullet B} = \bar{A} + \bar{B}$$



$$\overline{A + B} = \bar{A} \bullet \bar{B}$$



Proof of DeMorgan's Laws (part 1)

- Show the truth table of left and right side match

$$\overline{x+y} = \bar{x} \cdot \bar{y}$$

x	y	$x+y$	$\overline{x+y}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

x	y	$\bar{x}$	$\bar{y}$	$\overline{xy}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

Proof of DeMorgan's Laws (part 2)

- Show the truth table of left and right side match $\overline{x \cdot y} = \bar{x} + \bar{y}$

x	y	xy	$\overline{xy}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

x	y	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

Useful Theorems

- **Minimization**
  - $XY + \bar{X}Y = Y$
  - $(X + Y)(\bar{X} + Y) = Y$
- **Absorption**
  - $X + XY = X$
  - $X(X + Y) = X$
- **Simplification**
  - $X + \bar{X}Y = X + Y$
  - $X(\bar{X} + Y) = XY$
- **Consensus**
  - $AB + \bar{A}C + BC = AB + \bar{A}C$
  - $(A + B)(\bar{A} + C)(B + C) = (A + B)(\bar{A} + C)$

Variables and Literals

- $F = xy + x\bar{y}z$
- There are **3 variables:** $\mathbf{x, y, z}$
- There are **5 literals:** $\mathbf{x, y, x, \bar{y}, z}$
  - Includes complements and duplicates

Expression Simplification

- Simplify to contain the smallest number of literals.

Example: $\mathbf{AB + \overline{A}CD + \overline{A}BD + \overline{A}C\overline{D} + ABCD}$

$= AB + ABCD + \overline{A}CD + \overline{A}C\overline{D} + \overline{A}BD$
$= AB(1 + CD) + \overline{A}C(D + \overline{D}) + \overline{A}BD$ (distributive)
$= AB + \overline{A}C(D + \overline{D}) + \overline{A}BD$ (Null Element)
$= AB + \overline{A}C + \overline{A}BD$ (Complement)
$= B(A + \overline{A}D) + \overline{A}C$ (distributive, commutative)
$= B(A + D) + \overline{A}C$ (Simplification Theorem)

How many literals in answer? 5

Complementing Functions

- Sometime useful for a circuit to use more AND then OR gates or vice versa
- Use DeMorgan’s
- Example: Complement $\mathbf{F} = \bar{\mathbf{x}}\,\mathbf{y}\,\mathbf{z} + \mathbf{x}\,\bar{\mathbf{y}}\,\bar{\mathbf{z}}$

$$\bar{\mathbf{F}} = \overline{\bar{\mathbf{x}}\,\mathbf{y}\,\mathbf{z} + \mathbf{x}\,\bar{\mathbf{y}}\,\bar{\mathbf{z}}}$$

$$= \left(\overline{\bar{\mathbf{x}}\,\mathbf{y}\,\mathbf{z}}\right)\left(\overline{\mathbf{x}\,\bar{\mathbf{y}}\,\bar{\mathbf{z}}}\right)$$

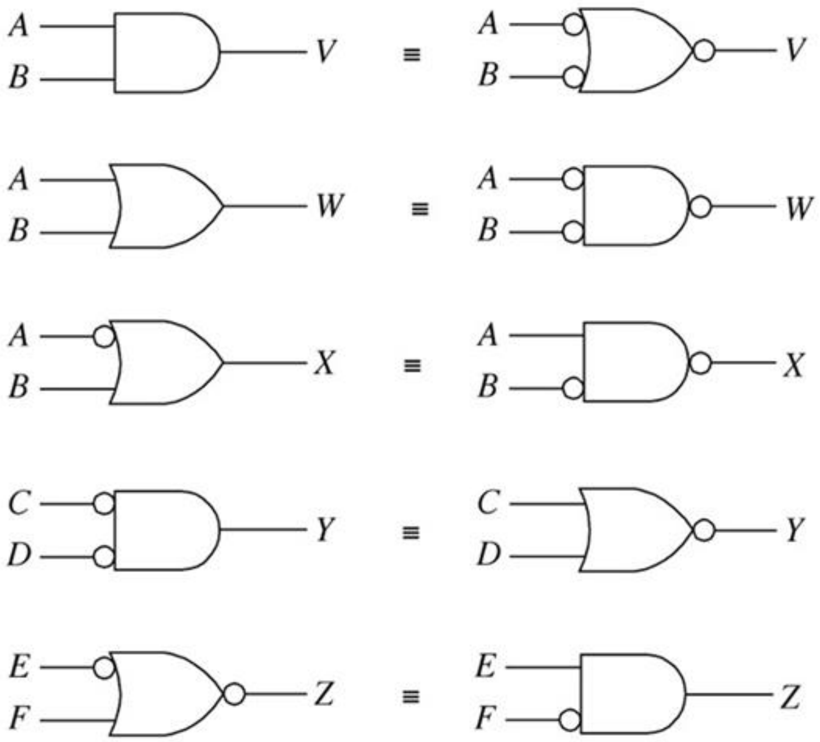
$$= \left(\mathbf{x} + \bar{\mathbf{y}} + \mathbf{z}\right)\left(\bar{\mathbf{y}} + \mathbf{y} + \mathbf{z}\right)$$

Complementing Circuits

- Bubble Pushing

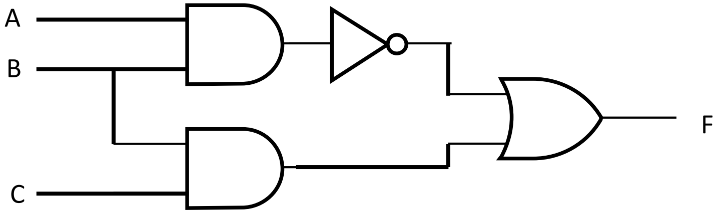
Steps:

1. Place bubble on output of circuit
2. Push bubbles from right to left (output to input) until all bubbles have been pushed to the inputs.
3. When pushing bubbles through a gate, add bubbles where there were none, and remove where there are. Then change gate from AND to OR and OR to AND.

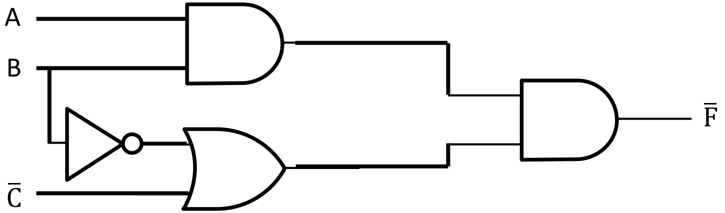


Bubble Pushing Example

Complement F using bubble pushing



Answer



Algebraic check:

$$F = \overline{AB} + BC$$

$$\bar{F} = \overline{\overline{AB} + BC}$$

$$= \left(\overline{\overline{AB}}\right)\left(\overline{BC}\right)$$

$$= (AB)(\bar{B} + \bar{C})$$

Duals and Self Dual

- A **dual** is an expression in which the
  - OR and AND are exchanged and the
  - 1s and 0s are exchanges
- A **self dual** is when $\mathrm{F} = \mathrm{F}^{\mathrm{D}}$ where F^{D} is the dual of F.

Self Dual example

- Example: Is $\mathrm{ab + bc + ac}$ self dual?

$\mathrm{ab + bc + ac}$

$= \overline{\overline{\mathrm{ab + bc + ac}}}$ (Involution)

$= \overline{\overline{ab} \cdot \overline{bc} \cdot \overline{ac}}$ (DeMorgan's)

$= \overline{(\bar{a} + \bar{b})(\bar{b} + \bar{c})(\bar{a} + \bar{c})}$ (DeMorgan's)

$= \overline{(\bar{a}\bar{b} + \bar{a}\bar{c} + \bar{b}\bar{b} + \bar{b}\bar{c})(\bar{a} + \bar{c})}$ (Distributive)

$= \overline{\bar{a}\bar{b}\bar{a} + \bar{a}\bar{c}\bar{a} + \bar{b}\bar{b}\bar{a} + \bar{b}\bar{c}\bar{a} + \bar{a}\bar{b}\bar{c} + \bar{a}\bar{c}\bar{c} + \bar{b}\bar{b}\bar{c} + \bar{b}\bar{c}\bar{c}}$ (Distributive)

$= \overline{\bar{a}\bar{b} + \bar{a}\bar{c} + \bar{a}\bar{b}\bar{c} + \bar{b}\bar{c}}$ (Indempotence)

$= \overline{\bar{a}\bar{b} + \bar{a}\bar{c} + \bar{b}\bar{c}}$ (Concensus theorem)

$= \overline{\bar{a}\bar{b}} \cdot \overline{\bar{a}\bar{c}} \cdot \overline{\bar{b}\bar{c}}$ (DeMorgan's)

$= (a + b)(a + c)(b + c)$ Yes, it is a dual

Summary

- Logic operators: AND, OR, NOT, NAND, NOR, XOR
- Plug in values to circuit or Boolean expression find truth table
- Boolean identities allow you to simplify Boolean algebras

References

- https://www.cs.ucr.edu/~ehwang/courses/cs120a/00winter/minterms.pdf

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