Canonical Logic Forms

CMSC 313
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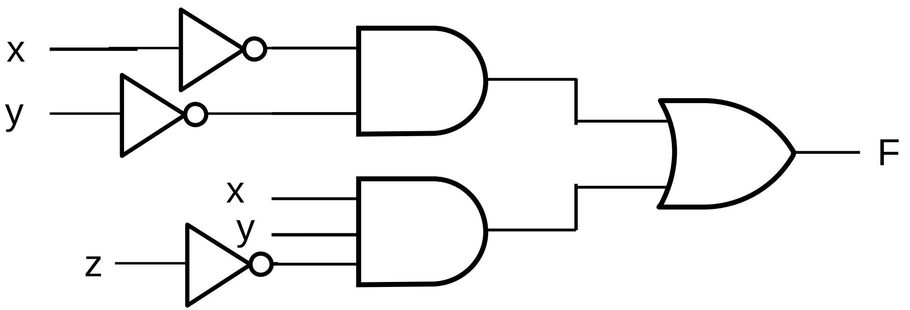
Converting between expression and circuit

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

$$F = \bar{x}\bar{y} + xy\bar{z}$$

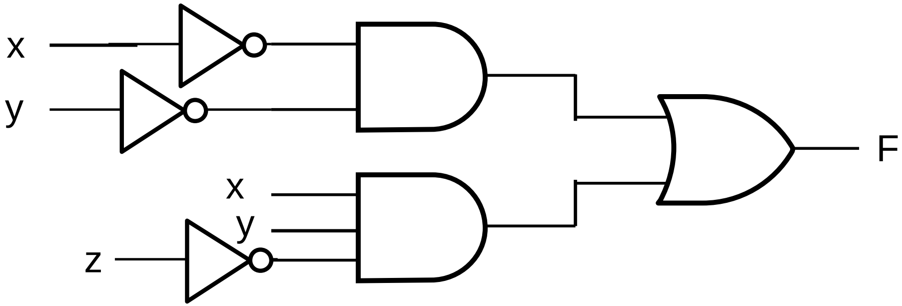
Translation



Converting to truth table: **plug in values!**

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

$F = \bar{x}\bar{y} + xy\bar{z}$



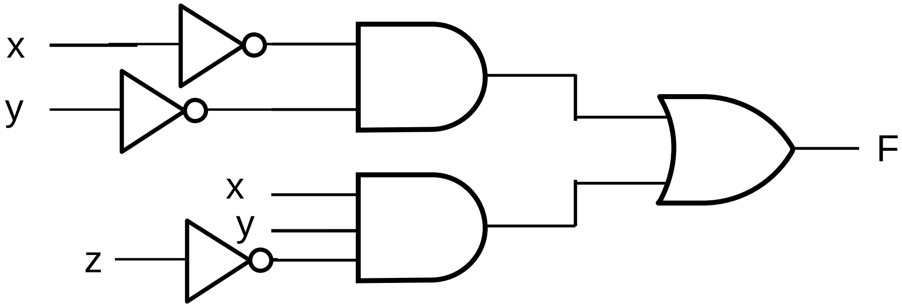
How to convert the other way?

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

?

$F = \bar{x}\bar{y} + xy\bar{z}$

?



Canonical Forms

- Boolean expressions that have a consistent form
- Each expression has a one to one correlation to a truth table
- Two kinds:
  - Sum of **minterms**, Sum of Products (SOP)
  - Product of **Maxterms**, Product of Sums (POS)
- Circuits that are shallower: logic has to pass through fewer circuits from input to output. This is faster because of gate delay.

minterm

- A **minterm**, denoted as $\mathbf{m_i}$, where $0 \le i < 2^n$, is a product (AND) of the n variables in which each variable is
    - **complemented** if the value assigned to it is **1**, and
    - **uncomplemented** if it is **0**.
- m_i is associated with the **ith row** out of n rows in the truth table
- Any Boolean function can be expressed as a sum (OR) of its minterms.
- A sum of minterms is called Sum of Products (SOP)

minterms of 3 variables

- A shorthand notation:
  F(list of variables) = Σ(list of 1-minterm indices)

- Example:

$$F = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz$$
$$= m_3 + m_5 + m_6 + m_7$$
$$= \sum(3,5,6,7)$$

x	y	z	minterm	notation
0	0	0	$\bar{x}\bar{y}\bar{z}$	m_0
0	0	1	$\bar{x}\bar{y}z$	m_1
0	1	0	$\bar{x}y\bar{z}$	m_2
0	1	1	$\bar{x}yz$	m_3
1	0	0	$x\bar{y}\bar{z}$	m_4
1	0	1	$x\bar{y}z$	m_5
1	1	0	$xy\bar{z}$	m_6
1	1	1	xyz	m_7

Inverse of minterm

- The inverse of a sum of minterms is a sum of all the remaining minterms
- DeMorgan's application can be complicated
- Example: find inverse $\bar{F}$

$$F = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz$$
$$= m_3 + m_5 + m_6 + m_7$$
$$= \sum(3,5,6,7)$$

$$\bar{F} = m_0 + m_1 + m_2 + m_4$$
$$= \sum(0,1,2,4)$$

x	y	z	minterm	F	F'
0	0	0	$\bar{x}\bar{y}\bar{z} = m_0$	0	1
0	0	1	$\bar{x}\bar{y}z = m_1$	0	1
0	1	0	$\bar{x}y\bar{z} = m_2$	0	1
0	1	1	$\bar{x}yz = m_3$	1	0
1	0	0	$x\bar{y}\bar{z} = m_4$	0	1
1	0	1	$x\bar{y}z = m_5$	1	0
1	1	0	$xy\bar{z} = m_6$	1	0
1	1	1	$xyz = m_7$	1	0

Convert expression into SOP using a truth table

Example: $F = x + yz$

Step 1. Derive truth table

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

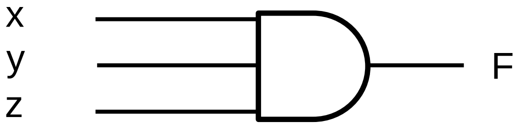
Step 2. Derive SOP

$$F = m_3 + m_4 + m_5 + m_6 + m_7$$

$$= \sum(3,4,5,6,7)$$

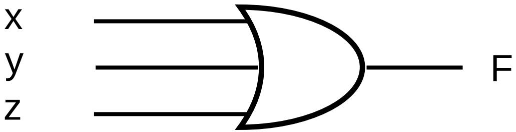
Multiple Input Gates

- For AND gates with input set S, if all elements in S equal 1, then output is 1. Otherwise the output is 0.
- For OR gates with input set S, if all elements in S equal 0, then output is 0. Otherwise the output is 1.



3-in AND

X	Y	Z	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1



3-in OR

X	Y	Z	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Convert SOP into circuit

Example: $F = m_2 + m_3 + m_7$ (for 3 variables)

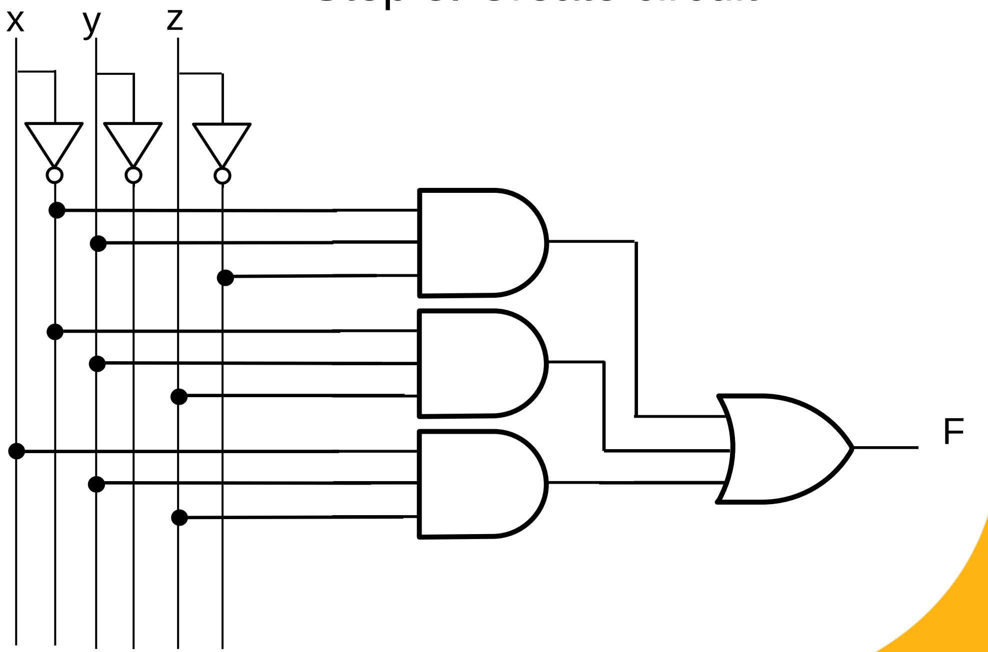
Step 1. Derive truth table

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Step 2. Extract minterms

$F = \bar{x}y\bar{z} + \bar{x}yz + xyz$

Step 3. Create circuit



Maxterm

- A **Maxterm**, denoted as M_i, where $0 \leq i < 2^n$, is a sum (OR) of the n variables (literals) in which each variable is
  - **complemented** if the value assigned to it is **1**, and
  - **uncomplemented** if it is **0**.
  - Note this is reverse of the definition for minterms
- Any Boolean function can be expressed as a product (AND) of its Maxterms.
- A product of Maxterms is called Products of Sums (POS)

Maxterms of 3 variables

- A shorthand notation:
  F(list of variables) = Π(list of Maxterm indices)
- Π is read “product of”

- Example: find Π notation of:
  $$F = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)(\bar{x}+y+z)$$
  $$= M_0M_1M_2M_4$$
  $$= \prod(0,1,2,4)$$

x	y	z	Maxterm	notation
0	0	0	$x+y+z$	M_0
0	0	1	$x+y+\bar{z}$	M_1
0	1	0	$x+\bar{y}+z$	M_2
0	1	1	$x+\bar{y}+\bar{z}$	M_3
1	0	0	$\bar{x}+y+z$	M_4
1	0	1	$\bar{x}+y+\bar{z}$	M_5
1	1	0	$\bar{x}+\bar{y}+z$	M_6
1	1	1	$\bar{x}+\bar{y}+\bar{z}$	M_7

Maxterms of the **zeros** are the output!

$$F = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)(\bar{x}+y+z)$$

$$= M_0 M_1 M_2 M_4$$

$$= \prod(0,1,2,4)$$

x	y	z	Maxterm	notation	F
0	0	0	$x+y+z$	M_0	0
0	0	1	$x+y+\bar{z}$	M_1	0
0	1	0	$x+\bar{y}+z$	M_2	0
0	1	1	$x+\bar{y}+\bar{z}$	M_3	1
1	0	0	$\bar{x}+y+z$	M_4	0
1	0	1	$\bar{x}+y+\bar{z}$	M_5	1
1	1	0	$\bar{x}+\bar{y}+z$	M_6	1
1	1	1	$\bar{x}+\bar{y}+\bar{z}$	M_7	1

Inverse of Maxterm

- The inverse of a product of Maxterms is a product of all the remaining Maxterms
- Example: Find $\bar{F}$

$$F = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)(\bar{x}+y+z)$$
$$= M_0 M_1 M_2 M_4$$
$$= \prod(0,1,2,4)$$

$$\bar{F} = M_3 M_5 M_6 M_7$$
$$= \prod(3,5,6,7)$$

x	y	z	Maxterm	F	F’
0	0	0	$x+y+z = M_0$	0	1
0	0	1	$x+y+\bar{z} = M_1$	0	1
0	1	0	$x+\bar{y}+z = M_2$	0	1
0	1	1	$x+\bar{y}+\bar{z} = M_3$	1	0
1	0	0	$\bar{x}+y+z = M_4$	0	1
1	0	1	$\bar{x}+y+\bar{z} = M_5$	1	0
1	1	0	$\bar{x}+\bar{y}+z = M_6$	1	0
1	1	1	$\bar{x}+\bar{y}+\bar{z} = M_7$	1	0

Convert expression into POS using a truth table

Example: $F = x + yz$

Step 1. Derive truth table

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Step 2. Derive POS

$$F = M_0 M_1 M_2$$

$$= \prod(0,1,2)$$

minterm and Maxterm

Example

F:

$$F = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz = m_3 + m_5 + m_6 + m_7 = \sum(3,5,6,7)$$

$$F = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)(\bar{x}+y+z) = M_0M_1M_2M_4 = \prod(0,1,2,4)$$

$\bar{F}$:

$$\bar{F} = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z} = m_0 + m_1 + m_2 + m_4 = \sum(0,1,2,4)$$

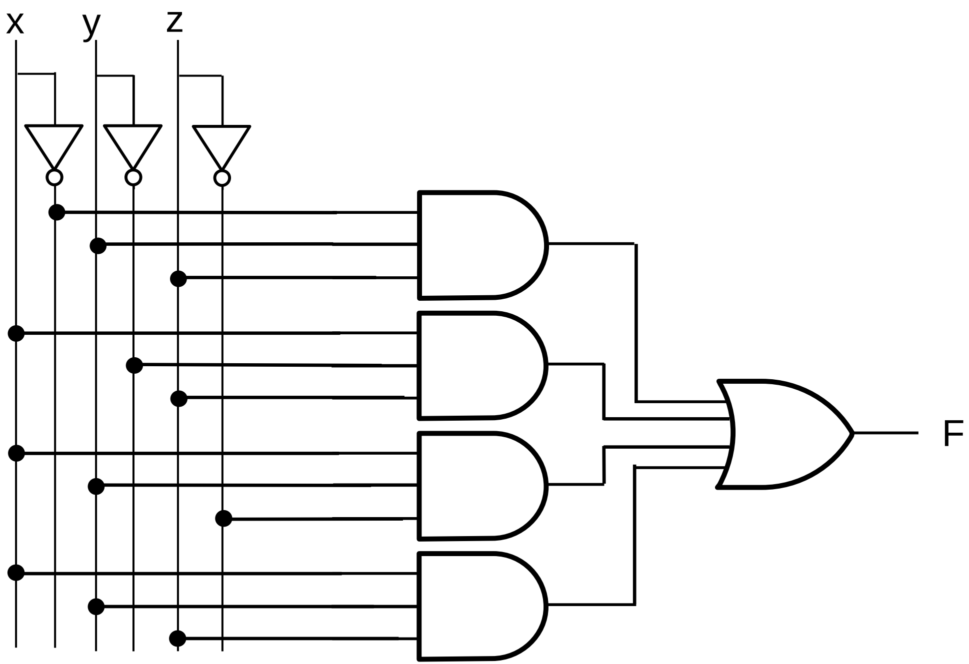
$$\bar{F} = (x+\bar{y}+\bar{z})(\bar{x}+y+\bar{z})(\bar{x}+\bar{y}+z)(\bar{x}+\bar{y}+\bar{z}) = M_3M_5M_6M_7 = \prod(3,5,6,7)$$

Are these (non-canonical) POS, SOP, both, neither?

$\bar{a}b + cd$	SOP
$c + \bar{a}$	SOP and POS
$(c + \bar{a})(d + \bar{a} + b)$	POS
$(c + \bar{a})d$	POS
$(c + \bar{a})db$	POS
$(c + \bar{a})(db + \bar{a})$	neither

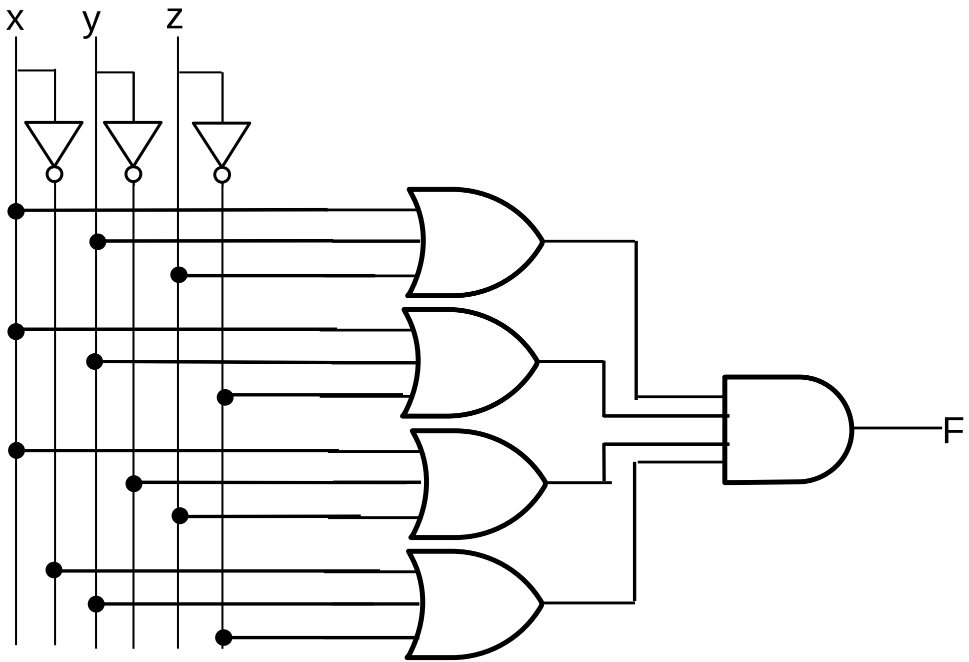
Circuits from minterms

$$F = m_3 + m_5 + m_6 + m_7$$
$$= \sum(3,5,6,7)$$



x	y	z	minterm	F	F'
0	0	0	$\bar{x}\bar{y}\bar{z} = m_0$	0	1
0	0	1	$\bar{x}\bar{y}z = m_1$	0	1
0	1	0	$\bar{x}y\bar{z} = m_2$	0	1
0	1	1	$\bar{x}yz = m_3$	1	0
1	0	0	$x\bar{y}\bar{z} = m_4$	0	1
1	0	1	$x\bar{y}z = m_5$	1	0
1	1	0	$xy\bar{z} = m_6$	1	0
1	1	1	$xyz = m_7$	1	0

Circuits from Maxterms

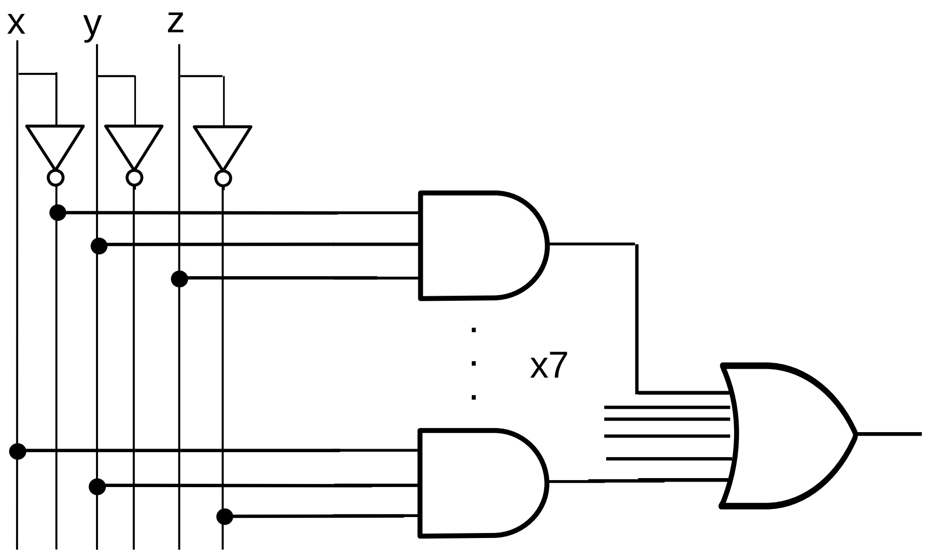


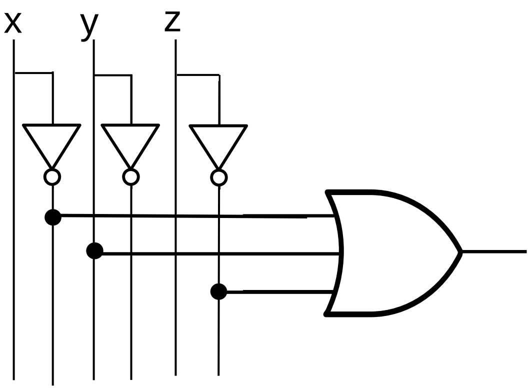
x	y	z	Maxterm	F	F’
0	0	0	$x + y + z = M_0$	0	1
0	0	1	$x + y + \bar{z} = M_1$	0	1
0	1	0	$x + \bar{y} + z = M_2$	0	1
0	1	1	$x + \bar{y} + \bar{z} = M_3$	1	0
1	0	0	$\bar{x} + y + z = M_4$	0	1
1	0	1	$\bar{x} + y + \bar{z} = M_5$	1	0
1	1	0	$\bar{x} + \bar{y} + z = M_6$	1	0
1	1	1	$\bar{x} + \bar{y} + \bar{z} = M_7$	1	0

$$F = M_0 M_1 M_2 M_4$$
$$= \prod (0,1,2,4)$$

minterms vs Maxterms

- For n variables, if POS uses x terms, SOP will use 2^n-x terms.
- Tradeoff means simpler circuit, cheaper to manufacture
- Example: $\sum(0,1,2,3,4,6,7) = \prod(5)$. POS is much simpler.





Self Duals

- Reminder: a Boolean expression is **self dual** if it equals its dual. A dual is produced by replacing all ANDs with ORs and vice versa and 1s with 0s.
- New definition: A Boolean expression is self dual if:

1. The expression is **neutral**, i.e. the number of minterms equals the number of Maxterms, and

2. The expression does **not** contain two **mutually exclusive** terms, e.g. xyz and $\bar{x}\bar{y}\bar{z}$ are mutually exclusive because all the variables in one term are complemented in the other. $x\bar{y}z$ and $\bar{x}y\bar{z}$ are also mutually exclusive.

Self dual example

- Is $a \oplus b$ self dual?

1. Is the expression neutral?

a	b	a⊕b
0	0	0
0	1	1
1	0	1
1	1	0

Yes, 2 minterms, 2 Maxterms

2. The expression contains **mutually exclusive** terms: $a \oplus b = a\bar{b} + \bar{a}b$, minterms m_1 and m_2 are mutually exclusive since the variables in $a\bar{b}$ are complemented in the other: $\bar{a}b$.

Not self dual

Self dual example

- Is $F = \sum(3,5,6,7)$ self-dual?

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

F is an inverted mirror. Therefore, there are an equal number of minterms and maxterms, and no terms are mutually exclusive. Therefore, F **is self dual**.

Simplest form

- Is either a minterm SOP or a maxterm POS the expression with the fewest literals? The simplest expression?
- No!
- Karnaugh maps are used to find the simplest expression and therefore a minimal literals and gates

Summary

- Canonical Form used to convert truth table to consistent expression
- Sum of **minterms**, Sum of Products (SOP)
- Product of **Maxterms**, Product of Sums (POS)
- SOP and POS have inverse quantity of terms

References

- https://www.cs.ucr.edu/~ehwang/courses/cs120a/00winter/minterms.pdf